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Extended Gauge Invariance in Geometrical Particle Models and the Geometry of W-Symmetry

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ABSTRACT

We prove that particle models whose action is given by the integrated n-th curvature function over the world line possess $n+1$ gauge invariances. A geometrical characterization of these symmetries is obtained via Frenet equations by rephrasing the n-th curvature model in $\mathbf{R}^d$ in terms of a standard relativistic particle in S^{d-n}. We "prove by example" that the algebra of these infinitesimal gauge invariances is nothing but W_{n+2}, thus providing a geometrical picture of the W-symmetry for these models. As a spin-off of our approach we give a new global invariant for four-dimensional curves subject to a curvature constraint.

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It is an amusing fact that the simplest of all possible invariants of curves in Minkowski space, *i.e.* the proper time, provides us with a nontrivial action for a relativistic particle. Upon (second) quantization, this system is described by a Klein-Gordon field, thus providing us with a particle description of a spin-zero field. It is also well known how to extend this formalism to the Dirac spinor by the introduction of world-line supersymmetry. Therefore, it is no less surprising that so little is known about geometrical particle actions associated with different geometrical invariants.

It has not been until recently that some of these actions have attracted considerable attention. The original motivation stemmed from the work of Polyakov [1] on rigid strings as a viable stringy description of QCD in four dimensions. Shortly thereafter, particle actions depending on the curvature of the world-line were studied as toy models for the string case [2]. But it was soon realised that some of these models had interesting properties of their own [3]. In particular, it was shown by Plyushchay [4] that the Minkowskian "rigid particle" action given by

$$S = \alpha \int ds \, \kappa, \tag{1.1}$$

where α is a dimensionless coupling constant, and the extrinsic curvature κ is given by

$$\kappa^2 = \eta_{\mu\nu} \frac{d^2 x^\mu}{ds^2} \frac{d^2 x^\nu}{ds^2} \, , \tag{1.2}$$

was upon (first) quantization a potential candidate for the description of photons and higher spin fields. Furthermore, this model appeared to have an unexpected extra gauge symmetry in addition to standard worldline reparametrizations [5]. Little was known, however, about the nature of this new gauge invariance. It was not until recently that an understanding of its algebraic aspects was unveiled due to an unexpected connection between this model and integrable systems of the KdV-type. It was shown in [6] that the equation of motion coming from (1.1) can be recast in terms of the Boussinesq Lax operator. By using standard methods in integrable hierarchies, this relationship allows the invariances of the model to be displayed explicitly, and shows that their algebra is nothing but a classical version of Zamolodchikov's W₃-algebra [7]. The interest of this result is twofold, on the one hand it brings together two previously unrelated fields –particle models and extended conformal algebras– on the other hand it offers a natural basis for the study of the geometry behind W-symmetry, a topic which has been recently investigated from several different viewpoints [8][9] . However, a complete understanding of the geometry behind the extended gauge invariance of the model (1.1) was still lacking in the analysis of [6].

The main purpose of this present work is to clarify all the geometrical aspects behind this extended symmetry. In the process it will become obvious how to extend the results in the "rigid" particle case to particle models whose actions are given by integrated curvature functions of higher order. By a judicious use of Frenet equations in d-dimensional Euclidean

space[1] we will be able to map the problem of a particle whose action is given by

$$S_n = \alpha_n \int_\gamma ds \; \kappa_n, \tag{1.3}$$

with κ_n its n-th curvature function, into the problem of a standard relativistic particle moving on the $(d-n)$-sphere. This equivalence will enable us to prove that the action (1.3) enjoys $n+1$ gauge symmetries, and moreover to give a geometrical characterization of them.

The plan of the paper is as follows:

In Section 2, in order to make the paper as self-contained as possible, we introduce the basic concepts about the geometry of curves in $\mathbf{R}^d$ that are needed in the following sections.

In Section 3 we work out the Hamiltonian analysis for the first simple cases, *i.e.* the standard relativistic particle and the models associated with the first and second curvatures, and we introduce the required geometrical tools as needed. In particular, we show how the extended gauge invariance of those models can be identified with Virasoro, W_3, and W_4 respectively, and how to interpret them geometrically.

In Section 4 we generalize the previous methods to the case of arbitrarily high curvature functions to show that the n-th curvature model can be rephrased in a way that is manifestly invariant under $n+1$ gauge symmetries.

Section 5 is a "divertimento" about global invariants of curves. We discuss how these particles models can provide us with new invariants of curves, and explicitly display one of them for the case of curves in four dimensions subject to the curvature constraint $\kappa_1 = \kappa_3$.

We conclude with some comments about possible generalizations of these methods to the string case, and a possible Lie algebraic classification of geometrical particle models in $\mathbf{R}^n$ with extended gauge invariance.

§2   A Peek at the Geometry of curves in $\mathbf{R}^d$

The purpose of this section is to give a simple and general introduction to the various geometrical constructions that will be needed later, and to set up the notation for the rest of the paper.

We will be working in $\mathbf{R}^d$, and we will assume the standard Euclidean metric, although the contents of this section are easily generalized to arbitrary Riemmanian metrics (see for example [**10**]). Our basic object of study will be the geometry of curves

$$\begin{aligned} \gamma : \; [t_0, t_1] & \;\rightarrow\; & \mathbf{R}^d \\ t & \;\mapsto\; & \mathbf{x}(t), \end{aligned} \tag{2.1}$$

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[1] From now on we will work in Euclidean space to enjoy the benefits of a positive definite metric.

which are immersions, *i.e.* which satisfy $\dot{\mathbf{x}}(t) = d\mathbf{x}/dt \neq 0$ for all $t \in [t_0, t_1]$. It will be useful in what follows to consider arc-length parametrized curves. The arc-length function $s : [t_0, t_1] \to \mathbf{R}$ is defined as usual by

$$s(t) = \int_{t_0}^{t} dt' \; |\dot{\mathbf{x}}(t')|. \tag{2.2}$$

A tangent vector field to γ for all t is given by its velocity vector $\dot{\mathbf{x}}$. Let us now define the induced *einbein, e,* as the modulus of the above vector. We can now introduce the normalized tangent vector $\mathbf{v}_1$

$$\mathbf{v}_1 = \frac{1}{e}\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}. \tag{2.3}$$

Higher order curvature functions can now be recursively defined as follows. Since $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$ it follows that

$$0 = \frac{1}{2}\frac{d}{ds}(\mathbf{v}_1 \cdot \mathbf{v}_1) = \mathbf{v}_1 \cdot \frac{d\mathbf{v}_1}{ds}, \tag{2.4}$$

and therefore the vector $d\mathbf{v}_1/ds$ is everywhere perpendicular to $\mathbf{v}_1$. We now define the first curvature function, κ_1, as its modulus. If $\kappa_1(s) \neq 0$ for all s, we define $\mathbf{v}_2$ as the normalization of $\dot{\mathbf{v}}_1$. We then have

$$\frac{d\mathbf{v}_1}{ds} = \kappa_1 \mathbf{v}_2, \tag{2.5}$$

so that $\mathbf{v}_2$ is a unit vector along γ everywhere perpendicular to $\mathbf{v}_1$.

If we now take derivatives of the expressions $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, we obtain as before that the vector

$$\frac{d\mathbf{v}_2}{ds} + \kappa_1 \mathbf{v}_1 \tag{2.6}$$

is everywhere perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$, and once again, we define the second curvature function, κ_2, as its modulus. If we now define $\mathbf{v}_3$ as the unitary vector in the direction of (2.6) it trivially follows that

$$\frac{d\mathbf{v}_2}{ds} = \kappa_2 \mathbf{v}_3 - \kappa_1 \mathbf{v}_1. \tag{2.7}$$

It can be now shown inductively [10] that if we have a set of orthonormal vector fields $\mathbf{v}_1, ..., \mathbf{v}_i$ along γ and $\kappa_1, ..., \kappa_{i-1}$ are nowhere zero curvature functions, then[2]

$$\frac{d\mathbf{v}_{i-1}}{ds} = \kappa_{i-1}\mathbf{v}_i - \kappa_{i-2}\mathbf{v}_{i-2}, \tag{2.8}$$

and moreover that

$$\frac{d\mathbf{v}_i}{ds} + \kappa_{i-1}\mathbf{v}_{i-1} \tag{2.9}$$

is a vector everywhere perpendicular to $\mathbf{v}_1, ..., \mathbf{v}_i$. If $\kappa_1, ..., \kappa_{i-1}$ are nowhere zero and κ_i is identically zero the set $\mathbf{v}_1, ..., \mathbf{v}_i$ define a "Frenet frame" for the curve γ, and the system of equations (2.8) are customarily called "Frenet equations".

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[2] Notice that the following formula is also valid for all i with the proviso that $\mathbf{v}_j = 0$ for $j \leq 0$.

The geometrical importance of the curvature functions is revealed by the fact that they provide a complete set of invariants for curves in Euclidean spaces. More explicitly,

• If $\kappa_i = 0$ this implies that all functions κ_j with $j > i$ are also zero.

• If a curve has nowhere vanishing curvatures up to κ_{i-1} and κ_i is identically zero then the curve lives in an i-dimensional hyperplane.

• Two parametrized curves in $\mathbf{R}^d$, γ and $\tilde{\gamma}$, such that $\tilde{e} = e$ and $\tilde{\kappa}_i = \kappa_i$ for $i = 1, ..., d$ are equivalent up to Euclidean motions, *i.e.* global translations and rotations.

Although from the results above we have an algorithm to construct arbitrarily high curvature functions, we will need, in order to do the Hamiltonian analysis of the associated actions, the explicit formulas for general κ_n in terms of derivatives of $\mathbf{x}$ in an arbitrary parametrization.

Let us define recursively the vector $\mathbf{x}_\perp^{(i)}$ as

$$\mathbf{x}_\perp^{(i)} = \mathbf{x}^{(i)} - \sum_{j=1}^{i-1} \frac{\mathbf{x}^{(i)} \cdot \mathbf{x}_\perp^{(j)}}{(\mathbf{x}_\perp^{(j)})^2} \mathbf{x}_\perp^{(j)}, \tag{2.10}$$

i.e. $\mathbf{x}_\perp^{(i)}$ is given by the projection of $\mathbf{x}^{(i)}$ onto the subspace which is orthogonal to all derivatives of $\mathbf{x}$ up to order $(i-1)$.

The orthogonality conditions on $\mathbf{x}_\perp^{(i)}$ imply in particular

$$\frac{d\mathbf{x}_\perp^{(i)}}{dt}\mathbf{x}_\perp^{(j)} = -\mathbf{x}_\perp^{(i)}\frac{d\mathbf{x}_\perp^{(j)}}{dt} = 0, \qquad j < i - 1. \tag{2.11}$$

Accordingly, the vector $d\mathbf{x}_\perp^{(i)}/dt$ admits the following expansion

$$\frac{d\mathbf{x}_\perp^{(i)}}{dt} = \mathbf{x}_\perp^{(i+1)} + A\,\mathbf{x}_\perp^{(i)} + B\,\mathbf{x}_\perp^{(i-1)}. \tag{2.12}$$

The coefficients A and B can be readily determined by multiplying (2.12) by $\mathbf{x}_\perp^{(i)}$ and $\mathbf{x}_\perp^{(i-1)}$ respectively. We get

$$\frac{d\mathbf{x}_\perp^{(i)}}{dt} = \mathbf{x}_\perp^{(i+1)} + \left(\frac{d}{dt}\log\sqrt{(\mathbf{x}_\perp^{(i)})^2}\right)\mathbf{x}_\perp^{(i)} - \frac{(\mathbf{x}_\perp^{(i)})^2}{(\mathbf{x}_\perp^{(i-1)})^2}\,\mathbf{x}_\perp^{(i-1)}. \tag{2.13}$$

Now taking into account that

$$\mathbf{v}_j = \frac{\mathbf{x}_\perp^{(j)}}{\sqrt{\left(\mathbf{x}_\perp^{(j)}\right)^2}}, \tag{2.14}$$

it is a direct computation to check that consistency of (2.13) with Frenet equations implies

$$\kappa_j = \sqrt{\frac{\left(\mathbf{x}_\perp^{(j+1)}\right)^2}{\dot{\mathbf{x}}^2\left(\mathbf{x}_\perp^{(j)}\right)^2}}. \tag{2.15}$$

With all of this in mind we can now start the Hamiltonian analysis of the dynamical systems defined by (1.3).

§3 Gauge invariance in geometrical particle actions

The purpose of this section is to show that certain geometrical particle actions possess an enlarged set of gauge symmetries in addition to reparametrization invariance. To show it we will make extensive use of the Hamiltonian formalism for constrained dynamical systems. We will later give the geometrical interpretation of such extended symmetries.

Consider a particle moving on a flat Euclidean d-dimensional manifold with position coordinates parametrized by the arc-length, $\mathbf{x}(s)$. We can build geometrical actions by using any of the curvature functions κ_i introduced in the previous section,

$$S = \int ds \, F(\kappa_1, \ldots, \kappa_{n-1}), \tag{3.1}$$

which are all, by construction, reparametrization invariant actions.

In view of the coordinate expression of the curvatures (2.15), the Lagrangian L in (3.1) will depend on the n-th order derivative of $\mathbf{x}$. The equations of motion will then be generically of the order $2n$. Phase space requires now more degrees of freedom than the standard coordinates and momenta $(\mathbf{x}, \mathbf{p})$ [11]. In fact, for an n-th order derivative Lagrangian the Hamiltonian formulation requires introducing as many as n momenta $\mathbf{p}_a$, which are conjugate to the variables $\mathbf{x}^{(a-1)} = d^{(a-1)}\mathbf{x}/dt^{(a-1)}$, with canonical Poisson brackets

$$\{x^{(a-1)\mu}, p_{b\nu}\} = \delta^\mu_\nu \delta^a_b, \qquad a, b = 1, \ldots, n, \tag{3.2}$$

where the definition of the momenta is given by

$$\mathbf{p}_a = \sum_{b=0}^{n-a} (-1)^b \frac{d^b}{dt^b} \left(\frac{\partial L}{\partial \mathbf{x}^{(b+a)}} \right) = \frac{\partial L}{\partial \mathbf{x}^{(a)}} - \dot{\mathbf{p}}_{a+1}. \tag{3.3}$$

The Hamiltonian is then defined as

$$H = \sum_{a=1}^{n} \mathbf{p}_a \mathbf{x}^{(a)} - L. \tag{3.4}$$

Such proliferation of degrees of freedom can also be seen when we set up the variational problem. An arbitrary infinitesimal variation $\delta\mathbf{x}$ of S can be rewritten as

$$\delta S = - \int_{t_0}^{t_1} dt \, \dot{\mathbf{p}}_1 \delta\mathbf{x} + \sum_{a=1}^{n} \mathbf{p}_a \delta\mathbf{x}^{(a-1)} \Big|_{t_0}^{t_1}. \tag{3.5}$$

So $\delta S = 0$ requires not only the equations of motion $\dot{\mathbf{p}}_1 = 0$ to be satisfied with fixed endpoints, but also all derivatives $\mathbf{x}^{(a)}$ up to the order $n-1$ should be kept fixed at the endpoints.

The standard relativistic particle, $F = $ constant

In the simplest case, with $F = $ constant, the action S measures the total arc-length of the curve. It is, for Minkowski target space metric, the familiar free relativistic particle action,

$$S_0 = m \int ds = m \int dt \sqrt{\dot{\mathbf{x}}^2}. \tag{3.6}$$

In this case the variational problem is set up in the traditional way, *i.e.* the critical curve is defined as the one minimising the action, while keeping the endpoints at fixed positions $\mathbf{x}_0$ and $\mathbf{x}_1$. S_0 is thus extremised by straight lines joining these two points,

$$\mathbf{x}(s) = \mathbf{x}_0 + \frac{\mathbf{p}}{m} \, s, \tag{3.7}$$

where one should require $\mathbf{x}(s_1) = \mathbf{x}_1$, with $s_1^2 = (\mathbf{x}_1 - \mathbf{x}_0)^2$. This results precisely in the mass-shell condition

$$\mathbf{p}^2 - m^2 = 0. \tag{3.8}$$

Such condition emerges in the Hamiltonian analysis of (3.6) as a constraint generated by the reparametrization invariance of the theory. Using the notation of the previous section we can write

$$\mathbf{p} = m\mathbf{v}_1, \qquad \dot{\mathbf{p}} = m\kappa_1 \mathbf{v}_2 = \ddot{\mathbf{x}} - \frac{\dot{e}}{e}\dot{\mathbf{x}} = 0. \tag{3.9}$$

The einbein $e(t) = ds/dt$ has to be a strictly positive function but it is otherwise completely arbitrary. Via the redefinition $\mathbf{x} \to e^{1/2}\mathbf{x}$ we can rewrite the equations of motion as

$$\mathsf{L}_2 \, \mathbf{x} = \ddot{\mathbf{x}} + T\mathbf{x} = 0, \tag{3.10}$$

with

$$T = \frac{1}{2}\frac{\ddot{e}}{e} - \frac{3}{4}\frac{\dot{e}^2}{e^2}, \tag{3.11}$$

where L_2 is the Lax operator for the KdV equation. This is nothing but the first example of an infinite series of Lax operators associated with generalized n-th KdV-hierarchies, which are of the form:

$$\mathsf{L}_n\Psi = (\partial^n + V_1\partial^{n-2} + \ldots + V_{n-2}\partial + V_{n-1})\Psi = 0. \tag{3.12}$$

Although it is clear from the previous discussion that the symmetries of equation (3.10) are simply reparametrizations[3], this result fits in the general framework developed by Radul in [12] where he shows that the symmetries of an equation of the type $\mathsf{L}_n\Psi = 0$ are nothing but W_n, a result that will be heavily used in the following. By recasting the equations of motion of the relativistic particle in this way we have simply rephrased reparametrization invariance of the model as the W_2 symmetry of (3.10).

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[3] But notice that the representation of the algebra of reparametrizations on T is only projective.

The rigid particle, $F \propto \kappa_1$

We may ask whether additional gauge invariances arise for particular choices of the function F. This is known to be the case among actions depending only on the first curvature. For generic dependence of the form $F(\kappa_1)$ the model is just reparametrization invariant. A new gauge symmetry appears though for the particular case of linear dependence [4][5]

$$S_1 = \alpha_1 \int ds\, \kappa_1 = \alpha_1 \int dt \sqrt{\frac{\ddot{\mathbf{x}}_\perp^2}{\dot{\mathbf{x}}^2}}, \tag{3.13}$$

which is also invariant under rigid scale transformations $\mathbf{x} \to \lambda \mathbf{x}$. It was shown in [6] that these two gauge transformations give a realization of the W_3 symmetry algebra. Let us sketch here how the proof goes.

We will assume throughout the rest of the paper that the highest curvature κ_n in (3.1) and, consequently, all lower curvatures are nowhere zero functions. The reason to do so becomes clear by noticing that for the action (3.13) to make sense as a variational problem one has to consider paths with nowhere vanishing first curvature κ_1. The equations of motion would otherwise blow up due to the non-analytic dependence of the Lagrangian on $\ddot{\mathbf{x}}_\perp^2$.

The Lagrangian is now second order and the phase space is given by $(\mathbf{x}, \mathbf{p}_1; \dot{\mathbf{x}}, \mathbf{p}_2)$, with

$$\mathbf{p}_2 = \frac{\alpha_1}{e} \mathbf{v}_2, \qquad \mathbf{p}_1 = -\alpha_1 \kappa_2 \mathbf{v}_3. \tag{3.14}$$

The submanifold where dynamics takes place is determined by the first-class constraints

$$\begin{aligned}
\varphi_1 &= \mathbf{p}_2 \dot{\mathbf{x}} \approx 0, & \varphi_2 &= \mathbf{p}_2^2 - \alpha_1^2/\dot{\mathbf{x}}^2 \approx 0, \\
\varphi_3 &= \mathbf{p}_1 \dot{\mathbf{x}} \approx 0, & \varphi_4 &= \mathbf{p}_1 \mathbf{p}_2 \approx 0, \\
& & \varphi_5 &= \mathbf{p}_1^2 \approx 0.
\end{aligned} \tag{3.15}$$

The first four constraints in (3.15) reflect the mutual orthogonality of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, whereas φ_5 implies that the second curvature vanishes, $\kappa_2 = 0$.

The existence of two primary first-class constraints[4] , φ_1 and φ_2, indicates that two gauge transformations are present. The two gauge degrees of freedom are given by the curvatures e and κ_1.

In a generic n-th order derivative theory primary constraints count the number of degenerate directions in the Lagrangian Hessian matrix $\partial^2 L/\partial \mathbf{x}^{(n)} \partial \mathbf{x}^{(n)}$. The requirement of stability of these constraints upon evolution (i.e. $\dot{\varphi}_i \approx 0$) usually brings in new sets of constraints, which might turn some of the primary constraints into second-class. The actual number of independent gauge transformations is given by the number of primary constraints that remain first-class after the stabilization procedure [13].

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[4] Primary constraints are those arising from the definition of the "highest" momentum [11], $\mathbf{p}_2$ in this case.

Let us show the nature of this new gauge symmetry. In Euclidean space the constraint φ_5 implies $\mathbf{p}_1 \approx 0$. The constraint surface is then described by the scalar constraints φ_1, φ_2 together with the vectorial constraint $\psi_\mu \equiv p_{1\mu}$, which are still first-class constraints among themselves.

Taking into account the Lagrangian expression of the momenta (3.14), the new constraint $\psi_\mu \approx 0$ implies that motion takes place on a plane. But any curve satisfying this single requirement will already be a solution of the equations of motion $\dot{\mathbf{p}}_1 = 0$!

The equation $\mathbf{p}_1 = 0$ can be recast, after a local rescaling $\mathbf{x} \to e\kappa_1^{1/3}\mathbf{x}$, into a form which is explicitly invariant under W_3 transformations

$$\mathsf{L}_3\,\mathbf{x} = \dddot{\mathbf{x}} + T\dot{\mathbf{x}} + (W + \frac{\dot{T}}{2})\mathbf{x}, \tag{3.16}$$

where L_3 is the Lax operator for the Boussinesq equation and T and W are given in terms of e and κ_1:

$$T = 2\frac{\ddot{e}}{e} - 3\frac{\dot{e}^2}{e^2} - \frac{\dot{e}\dot{\kappa}_1}{e\kappa_1} + \frac{\ddot{\kappa}_1}{\kappa_1} - \frac{4}{3}\frac{\dot{\kappa}_1^2}{\kappa_1^2} + e^2\kappa_1^2, \tag{3.17}$$

$$W = -\frac{1}{6}\frac{\dddot{\kappa}_1}{\kappa_1} + \frac{5}{6}\frac{\dot{\kappa}_1\ddot{\kappa}_1}{\kappa_1^2} - \frac{20}{27}\frac{\dot{\kappa}_1^3}{\kappa_1^3} - \frac{2}{3}\kappa_1\dot{\kappa}_1 e^2$$
$$- \frac{5}{6}\frac{\dot{\kappa}_1^2\dot{e}}{\kappa_1^2 e} - \frac{1}{2}\frac{\dot{\kappa}_1\dot{e}^2}{\kappa_1 e^2} + \frac{1}{2}\frac{\ddot{\kappa}_1\dot{e}}{\kappa_1 e} + \frac{1}{6}\frac{\dot{\kappa}_1\ddot{e}}{\kappa_1 e}. \tag{3.18}$$

So W_3 is the gauge algebra underlying this extended symmetry. Let us give the geometrical explanation for the emergence of this extra symmetry. This will in turn provide us with a geometrical characterization of the way finite W transformations are realized in this model.

Given a parametrized curve γ in $\mathbf{R}^d$ it naturally induces a map from the interval $[t_0, t_1]$ into the $(d-1)$-sphere (Gauss map) given by the direction of the unit tangent vector at any point in the curve:

$$\begin{aligned}\Gamma_1 : [t_0, t_1] &\to S^{d-1}\\ t &\mapsto \mathbf{v}_1(t).\end{aligned} \tag{3.19}$$

We have on the unit sphere S^{d-1} a natural metric given by the induced metric from $\mathbf{R}^d$. Now we can think of Γ_1 as a new curve and parametrize it by its own arc-length $\theta_1(t)$ as follows:

$$\theta_1(t) = \int_{t_0}^t dt\,\left|\frac{d\mathbf{v}_1}{dt}\right| = \int_0^{s(t)} ds\,\kappa_1, \tag{3.20}$$

which precisely coincides with the action S_1. Therefore, the rigid particle action S_1 in $\mathbf{R}^d$ is equivalent to a free relativistic particle moving on the $(d-1)$-dimensional sphere!

This provides us with a geometrical picture of the way W_3 symmetry acts on this model. Consider the map G_1 which assigns to a curve γ in $\mathbf{R}^d$ its associated Gauss map Γ_1:

$$G_1 : \quad \gamma \quad \rightarrow \quad \Gamma_1. \tag{3.21}$$

This is not an invertible map. This is because, given a Gauss map curve $\mathbf{v}_1(t)$, we can only determine, using Frenet equations, the combinations $e\kappa_i$, but not the actual values of e and κ_i which determine the original curve.

It is clear that a curve $\tilde{\gamma}$ with einbein and curvatures

$$\tilde{e}(t) = \frac{1}{\sigma(t)}e(t), \qquad \tilde{\kappa}_i(t) = \sigma(t)\kappa_i(t), \tag{3.22}$$

where $\sigma(t)$ is a strictly positive function, has the same Gauss map as the one of γ. Notice also that the two curves are not related by a reparametrization since the curvatures κ_i, which are scalar under reparametrizations, are actually different for γ and $\tilde{\gamma}$.

It is obvious from (3.20) that the action S_1 is insensitive to such deformations. We conclude that *two curves belong to the same W gauge orbit provided that they share the same Gauss map*:

$$\gamma \sim \tilde{\gamma} \quad \Longleftrightarrow \quad G_1(\gamma) = G_1(\tilde{\gamma}). \tag{3.23}$$

With this piece of geometric information it is now clear that critical curves of S_1 are given by geodesics in S^{d-1}. And this clearly implies that solutions always lie on a plane (as can be easily visualized in the three dimensional case). This is precisely what we have found following our previous constraint analysis. This connects with a well-known result due to Fenchel [14] in the global theory of curves. For any closed curve in d-dimensional Euclidean space the *total first curvature* satisfies

$$\oint ds\, \kappa_1 \geq 2\pi, \tag{3.24}$$

where the inequality is saturated for all plane convex curves, *i.e.* non-self-intersecting curves which always lie on one side of their tangent lines, and for no other curves.

Higher order F: The W_4 example

Going back to the general form of F (3.1), it is natural to ask whether specific higher curvature theories also enjoy extended gauge symmetries, what their geometrical meaning is, and eventually, whether they are related to general W_n algebras.

Consider a generic model (3.1) depending on the first n curvatures of γ. Its Lagrangian will depend on derivatives of $\mathbf{x}(t)$ up to order $n+1$. Primary constraints are determined from the definition of $\mathbf{p}_{n+1} = \partial L / \partial \mathbf{x}^{(n+1)}$. It is easy to see that we may have at most $(n+1)$ primary constraints. In fact, if we want them to be first class among themselves this

condition determines their form up to a function $g(\kappa_1, \ldots, \kappa_{n-1})$,

$$\phi_1 = \mathbf{p}_{n+1}\dot{\mathbf{x}}, \quad \phi_2 = \mathbf{p}_{n+1}\ddot{\mathbf{x}}_\perp, \quad \cdots \quad \phi_n = \mathbf{p}_{n+1}\mathbf{x}_\perp^{(n)}, \quad \phi_{n+1} = \mathbf{p}_{n+1}^2 - \frac{g^2}{(\mathbf{x}_\perp^{(n)})^2}. \tag{3.25}$$

This set of constraints can only follow from a Lagrangian linear in the highest curvature κ_n. The simplest case is given by actions measuring the total n-th curvature of the curve:

$$S_n = \alpha_n \int ds \; \kappa_n, \tag{3.26}$$

where α_n is a dimensionless coupling constant.

Let us first, for the sake of simplicity, study the model S_2 depending on the second curvature. The features appearing in this case will help to give us a picture of the general case.

In an arbitrary parametrization the action takes the form:

$$S_2 = \alpha_2 \int dt \; \sqrt{\frac{\dddot{\mathbf{x}}_\perp^2}{\ddot{\mathbf{x}}_\perp^2}}. \tag{3.27}$$

Coordinates in phase space are given by $(\mathbf{x}, \mathbf{p}_1; \dot{\mathbf{x}}, \mathbf{p}_2; \ddot{\mathbf{x}}, \mathbf{p}_3)$ and the Lagrangian expression of the "highest" momentum $\mathbf{p}_3$ is:

$$\mathbf{p}_3 = \frac{\alpha_2}{e^2 \kappa_1}\mathbf{v}_3 = \frac{\alpha_2 \dddot{\mathbf{x}}_\perp}{\sqrt{\ddot{\mathbf{x}}_\perp^2 \, \dddot{\mathbf{x}}_\perp^2}}, \tag{3.28}$$

which gives rise to the following three primary constraints

$$\phi_1 = \mathbf{p}_3\dot{\mathbf{x}} \approx 0, \qquad \phi_2 = \mathbf{p}_3\ddot{\mathbf{x}}_\perp \approx 0, \qquad \phi_3 = \mathbf{p}_3^2 - \frac{\alpha_2^2}{\ddot{\mathbf{x}}_\perp^2} \approx 0. \tag{3.29}$$

So we may have in principal up to three gauge symmetries. Time evolution is only consistent on the submanifold determined by the constraints and is governed by the Hamiltonian

$$H = \mathbf{p}_1\dot{\mathbf{x}} + \mathbf{p}_2\ddot{\mathbf{x}} + \sum_{i=1}^{3} \lambda_i \phi_i. \tag{3.30}$$

Here the three coeficients λ_i may be considered, in principle, as arbitrary functions of time. If any of the primary constraints should become second-class during the stabilization procedure, the associated coefficient λ_i would take a definite canonical form [13].

The functions λ_i do have a definite Lagrangian form. From the Hamiltonian equation of motion for $\ddot{\mathbf{x}}$

$$\frac{d\ddot{\mathbf{x}}}{dt} = \{\ddot{\mathbf{x}}, H\} = \lambda_1 \dot{\mathbf{x}} + \lambda_2 \ddot{\mathbf{x}}_\perp + 2\lambda_3 \mathbf{p}_3, \tag{3.31}$$

we obtain the following expressions of the λ_i in terms of the einbein e and the curvature functions κ_1 and κ_2:

$$\lambda_1 = \frac{\dot{\mathbf{x}}\dddot{\mathbf{x}}}{\dot{\mathbf{x}}^2} = \frac{\ddot{e}}{e} - e^2 \kappa_1^2,$$

$$\lambda_2 = \frac{\ddot{\mathbf{x}}_\perp \dddot{\mathbf{x}}}{\ddot{\mathbf{x}}_\perp^2} = 3\frac{\dot{e}}{e} + \frac{\dot{\kappa}_1}{\kappa_1}, \tag{3.32}$$

$$\lambda_3 = \frac{1}{2\alpha_2}\sqrt{\ddot{\mathbf{x}}_\perp^2 \dddot{\mathbf{x}}_\perp^2} = \frac{1}{2\alpha_2} e^5 \kappa_1^2 \kappa_2.$$

The stabilization of ϕ_1, ϕ_2, ϕ_3 produces a plethora of new constraints[5] $\phi_i \approx 0$, with:

$$\phi_4 = \tilde{\mathbf{p}}_2 \dot{\mathbf{x}}, \qquad \phi_5 = \tilde{\mathbf{p}}_2 \ddot{\mathbf{x}}_\perp, \qquad \phi_6 = \tilde{\mathbf{p}}_2 \mathbf{p}_3,$$

$$\phi_7 = \mathbf{p}_1 \dot{\mathbf{x}}, \qquad \phi_8 = \mathbf{p}_1 \ddot{\mathbf{x}}_\perp, \qquad \phi_9 = \mathbf{p}_1 \mathbf{p}_3 + \tilde{\mathbf{p}}_2^2 - \frac{\alpha_2^2}{\dot{\mathbf{x}}^2}, \tag{3.33}$$

$$\phi_{10} = \mathbf{p}_1 \mathbf{p}_3, \qquad \phi_{11} = \tilde{\mathbf{p}}_2 \mathbf{p}_1,$$

$$\phi_{12} = \mathbf{p}_1^2,$$

where

$$\tilde{\mathbf{p}}_2 = \mathbf{p}_2 + \frac{\dot{\mathbf{x}}\ddot{\mathbf{x}}}{\dot{\mathbf{x}}^2}\mathbf{p}_3 = -\frac{\alpha_2 \kappa_3}{e\kappa_1}\mathbf{v}_4. \tag{3.34}$$

These are all first-class constraints. So, indeed, the model has three gauge symmetries and by virtue of (3.32) the einbein e and the curvatures κ_1 and κ_2 can be regarded as the pure gauge degrees of freedom of the theory.

The constraints ϕ_{10} and ϕ_{11} actually arise in a non-standard way. On the constraint submanifold determined by $\phi_1, \ldots, \phi_9$ we can write

$$\dot{\phi}_8 = 2\lambda_3 \mathbf{p}_1 \mathbf{p}_3, \qquad \dot{\phi}_9 = \mathbf{p}_1 \mathbf{p}_2 - \lambda_2 \mathbf{p}_1 \mathbf{p}_3. \tag{3.35}$$

In order to make these expressions vanish one might naively think of choosing $\lambda_3 = 0$ and $\lambda_2 = (\mathbf{p}_1 \mathbf{p}_2)/(\mathbf{p}_1 \mathbf{p}_3)$. This would imply that ϕ_2 and ϕ_3 become second-class and we would be left with reparametrizations as the only gauge symmetry. However, according to (3.32), λ_3 is proportional to κ_2, which cannot be zero as it would render the whole Hamiltonian analysis ill-defined. So the only consistent choice in (3.35) was to assume $\phi_{10} = \mathbf{p}_1 \mathbf{p}_3$ as a new constraint.

Most of the constraints (3.33) are simply identities when rewritten in purely Lagrangian terms. One can check that the only non-trivial constraints in this sense are ϕ_{10} and ϕ_{12}. The constraint ϕ_{10} forces κ_3 to be equal to κ_1 and ϕ_{12} implies $\kappa_4 = 0$. For the theory to make sense the target space dimension should then be at least $d = 4$, otherwise the third curvature κ_3 would vanish.

[5] We assume the target space dimension to be sufficiently large so that it has enough room to accomodate all these constraints.

Let us show that W_4 is the underlying symmetry algebra of this model. In complete analogy with the W_3 case, we have the constraint $\mathbf{p}_1^2 \approx 0$ which in Euclidean space implies the vectorial constraint $\mathbf{p}_1 \approx 0$. This new constraint is first-class with respect to all ϕ_i. The Lagrangian expression of $\mathbf{p}_1$ in the Lagrangian submanifold determined by $\kappa_3 = \kappa_1$ is proportional to $\mathbf{v}_5$. This new constraint forces the particle to move in a four dimensional hyperplane. So any curve satisfying this condition and $\kappa_1 = \kappa_3$ will already be a solution of the equations of motion $\dot{\mathbf{p}}_1 = 0$. After a local rescaling in $\dot{\mathbf{x}}$ the equation $\mathbf{p}_1 = 0$ can be rewritten in terms of a fourth order generalized KdV Lax operator (3.12) as $\mathsf{L}_4\dot{\mathbf{x}} = 0$, with coefficients V_i depending on the gauge degrees of freedom e, κ_1 and κ_2. Therefore its symmetry algebra is given precisely by W_4.

To understand the origin of these two extra symmetries we shall study the curve Γ_1 on S^{d-1} determined by the Gauss map. Γ_1 has its own set of Frenet equations which we can relate to those of γ. The tangent space of S^{d-1} at a point $\mathbf{v}_1$ is spanned by $(\mathbf{v}_2, \ldots, \mathbf{v}_d)$, $\mathbf{v}_2$ being the unit vector tangent to Γ_1. Since the metric in S^{d-1} is the induced metric from flat $\mathbf{R}^d$ we have to introduce covariant derivatives. The Levi-Civita covariant derivative of a vector in an isometrically immersed manifold is obtained by computing its covariant derivative in the immersion manifold (the standard derivative in $\mathbf{R}^d$ in the present case) and projecting it on the immersed manifold.

The covariant derivatives of $(\mathbf{v}_2, \ldots, \mathbf{v}_d)$ along the curve $\mathbf{v}_1(t)$ are thus given by

$$\frac{D_1\mathbf{v}_j}{ds} = \frac{d\mathbf{v}_j}{ds} - \left(\frac{d\mathbf{v}_j}{ds}\mathbf{v}_1\right)\mathbf{v}_1, \qquad j \geq 2. \tag{3.36}$$

Using the Frenet equations of γ we can rewrite them as

$$\begin{aligned}\frac{D_1\mathbf{v}_2}{ds} &= \kappa_2\mathbf{v}_3, \\ \frac{D_1\mathbf{v}_j}{ds} &= \frac{d\mathbf{v}_j}{ds} = \kappa_j\mathbf{v}_{j+1} - \kappa_{j-1}\mathbf{v}_{j-1}, \quad j > 2.\end{aligned} \tag{3.37}$$

These are precisely the Frenet equations of Γ_1. Now we would like to introduce the Gauss map induced by the curve Γ_1. It will be a map from $[t_0, t_1]$ to the sphere S^{d-2}. However, this second Gauss map Γ_2 cannot be simply given by the unit tangent vector to Γ_1, $\mathbf{v}_2(t)$, because it belongs to different tangent spaces for different values of t. We can identify these tangent spaces by parallel transport to a reference point t_r. Let us define $U(t_r, t)$ as follows

$$\begin{aligned}U(t_r, t): \; TS^{d-1}_{\mathbf{v}_1(t)} &\longrightarrow TS^{d-1}_{\mathbf{v}_1(t_r)} \\ \mathbf{u}(t) &\mapsto \mathbf{u}^{[1]}(t) = U(t_r, t)\mathbf{u}(t),\end{aligned} \tag{3.38}$$

where $\mathbf{u}^{[1]}(t)$ is obtained by parallel transport of $\mathbf{u}(t)$, with respect to the natural connection on S^{d-1}, along the curve $\mathbf{v}_1$ from t to t_r.

The second Gauss map can now be defined as

$$\begin{aligned}\Gamma_2: \; [t_0, t_1] &\rightarrow S^{d-2} \\ t &\mapsto \mathbf{v}_2^{[1]}(t),\end{aligned} \tag{3.39}$$

where S^{d-2} is the unit sphere in $TS^{d-1}_{\mathbf{v}_1(t_r)} \approx \mathbf{R}^{d-1}$.

It is easy to show that the dependence of Γ_2 on the choice of the reference point t_r is irrelevant. It only amounts to a global rotation of the curve on the $(d-2)$-sphere. Notice also that, due to the non-trivial holonomy of S^{d-1}, closed curves γ will in general be mapped into open curves in S^{d-2}.

For a generic vector field $\mathbf{u}(t)$ along $\mathbf{v}_1$ we have the relation

$$\frac{d\mathbf{u}^{[1]}}{dt}(t) = U(t_r, t)\frac{D_1\mathbf{u}}{dt}(t), \tag{3.40}$$

so the Frenet equations (3.37) can be rewritten in terms of $\mathbf{v}_i^{[1]}$ by substituting covariant by ordinary derivatives

$$\begin{aligned}
\frac{d\mathbf{v}_2^{[1]}}{ds} &= \kappa_2\mathbf{v}_3^{[1]}, \\
\frac{d\mathbf{v}_j^{[1]}}{ds} &= \kappa_j\mathbf{v}_{j+1}^{[1]} - \kappa_{j-1}\mathbf{v}_{j-1}^{[1]}, \quad j > 2.
\end{aligned} \tag{3.41}$$

In particular, now we can parametrize the curve Γ_2 by its own arc-length, given by the induced metric on the sphere S^{d-2}

$$\theta_2(t) = \int_{t_0}^t dt \left|\frac{d\mathbf{v}_2^{[1]}}{dt}\right| = \int_0^{s(t)} ds\, \kappa_2, \tag{3.42}$$

which coincides with the action S_2.

Consider the map G_2 from the original curve to its second Gauss map

$$G_2: \quad \gamma \quad \to \quad \Gamma_2. \tag{3.43}$$

Two curves with the same values of $e\kappa_i$ for $i \geq 2$ share the same image. On the other hand, the action S_2 only depends on Γ_2 and may be regarded as a free particle action on S^{d-2}.

We have thus a neat interpretation of the W_4 symmetry on this model as those deformations of curves which preserve the second Gauss map.

Frenet equations and Hamiltonian constraints

It is now possible to give a more geometrical flavour to the Hamiltonian constraints that appeared in our models by a straight-forward application of the Hamiltonian formalism. The way to do so is, once again, via Frenet equations.

Let us start with the rigid particle. In this case the Hamiltonian equations of motion are given by

$$\begin{aligned}
\ddot{\mathbf{x}} &= \lambda_2\mathbf{p}_2 + \lambda_1\dot{\mathbf{x}}, \\
\dot{\mathbf{p}}_2 &= -\mathbf{p}_1 - \lambda_1\mathbf{p}_2 - \frac{\alpha_1^2\lambda_2}{e^4}\dot{\mathbf{x}}, \\
\dot{\mathbf{p}}_1 &= 0.
\end{aligned} \tag{3.44}$$

If we now plug the Lagrangian expressions of the momenta given by (3.14) in the equations above, and we choose to parametrize the curve by arc-length, the equation for time evolution

of $\mathbf{p}_1$ gives

$$\frac{d}{ds}(\kappa_2 \mathbf{v}_3) = 0. \tag{3.45}$$

Compatibility of this equation with the Frenet equation for $\mathbf{v}_3$ clearly implies $\mathbf{v}_3 = 0$, or equivalently $\kappa_2 = 0$, which is the only nontrivial constraint in the Lagrangian variables. It is now a straight-forward exercise to check that the other two equations of motion are nothing but Frenet equations in two dimensions.

A similar analysis can be carried out, with a little more computational effort, for the second curvature theory. In this case consistency of the Hamiltonian equations of motion with Frenet equations implies that $\kappa_4 = 0$ and $\kappa_1 = \kappa_3$. Moreover, the Hamiltonian equations of motion are Frenet equations in S^3. They only give us information about the Gauss map of the curve! Nevertheless the original curve in Euclidean space can be reconstructed in a given gauge because of the constraint $\kappa_1 = \kappa_3$.

§4 GEOMETRICAL ANALYSIS OF THE n-TH CURVATURE MODEL

Up to now we have studied a few models where the linear dependence on the curvature implied that the theory enjoys extended gauge symmetries. Our analysis has relied initially on the constraint Hamiltonian analysis, and only later has been rephrased in a geometrical manner. The Hamiltonian analysis becomes increasingly tedious as we go to higher order models, so the only practical way to actually prove the existence of extended gauge symmetries in these models will be through a geometrical analysis.

In the previous section we have constructed a sequence of Gauss maps

$$\gamma \quad \rightarrow \quad \Gamma_1 \quad \rightarrow \quad \Gamma_2. \tag{4.1}$$

where Γ_2 is the curve described by $\mathbf{v}_2^{[1]}$ on S^{d-2}. From the equations (3.41) it follows that $\mathbf{v}_3^{[1]}$ is the tangent unit vector of Γ_2 and that $(\mathbf{v}_3^{[1]}, \ldots, \mathbf{v}_d^{[1]})$ span the tangent space $TS_{\mathbf{v}_2^{[1]}}^{d-2}$.

It is clear that the same procedure followed in the previous section in order to construct Γ_2 from Γ_1 can now be systematically applied to construct higher Gauss maps.

The n-th Gauss map will be given by

$$\begin{aligned}
\Gamma_n : \; [t_0, t_1] \quad &\rightarrow \quad S^{d-n} \\
t \quad &\mapsto \quad \mathbf{v}_n^{[n-1]}(t),
\end{aligned} \tag{4.2}$$

where $\mathbf{v}_n^{[n-1]}(t)$ has been obtained by parallel transport of $\mathbf{v}_n^{[n-2]}(t)$ along the curve $\mathbf{v}_{n-1}^{[n-2]}(t)$ from t to t_r, with respect to the natural connection in S^{d-n+1}. It satisfies the equations

$$\begin{aligned}
\frac{d\mathbf{v}_n^{[n-1]}}{ds} &= \kappa_n \mathbf{v}_{n+1}^{[n-1]}, \\
\frac{d\mathbf{v}_j^{[n-1]}}{ds} &= \kappa_j \mathbf{v}_{j+1}^{[n-1]} - \kappa_{j-1} \mathbf{v}_{j-1}^{[n-1]}, \quad j > n.
\end{aligned} \tag{4.3}$$

$$- \; 15 \; -$$

These relations allow us to write the length element in the $(d-n)$-sphere as

$$\theta_n(t) = \int_{t_0}^t dt \left| \frac{d\mathbf{v}_n^{[n-1]}}{dt} \right| = \int_0^{s(t)} ds \, \kappa_n, \tag{4.4}$$

which is, up to a constant, the action of the n-th curvature model

$$S_n = \alpha_n \int ds \, \kappa_n. \tag{4.5}$$

In consequence, two parametrized curves γ and $\tilde\gamma$ in $\mathbf{R}^d$ will be gauge equivalent if they have the same n-th Gauss map

$$\gamma \sim \tilde\gamma \quad \Longleftrightarrow \quad \Gamma_n(\gamma) = \Gamma_n(\tilde\gamma). \tag{4.6}$$

In this case, if γ and $\tilde\gamma$ are characterized by (e, κ_i) and $(\tilde e, \tilde\kappa_i)$ respectively, according to equations (4.3) they have to be related by

$$\begin{aligned}
\tilde e &= \frac{1}{\sigma_0(t)} \, e, \\
\tilde\kappa_i &= \sigma_i(t) \, \kappa_i, \qquad i < n, \\
\tilde\kappa_i &= \sigma_0(t) \, \kappa_i, \qquad i \geq n,
\end{aligned} \tag{4.7}$$

where $\sigma_i(t)$ are strictly positive but otherwise arbitrary functions. These transformations, together with worldline reparametrizations form a set of $n+1$ gauge transformations for S_n.

It is clear from (4.6) that the n-th curvature model contains the gauge symmetry of all models S_j with $j < n$. All of this is reminiscent of the mechanism of reduction from W_n into W_j for all $j < n$ introduced in [**15**]. But unfortunately we lack a general proof of the equivalence of these extended gauge invariances with W_n. Therefore we have to rely, for the time being, on the few examples in which a direct computation of the algebra of symmetries is still feasible.

§5   GLOBAL INVARIANTS FOR CURVES FROM HAMILTONIAN DYNAMICS

The purpose of this section is to show how standard Hamiltonian methods applied to geometrical particle models can shed new light on global invariants of curves in $\mathbf{R}^d$. But first a word of caution: It is, of course, well known that any two closed curves immersed in $\mathbf{R}^d$ with $d \geq 4$ can be continuosly deformed one into the other, therefore it seems that there is not much more to be said about the subject. We will show that this is not the case if we consider curves which are subject to curvature constraints, *i.e.* its associated curvature functions are not fully independent.

First, as a warm up, we will reproduce some well known global results for curves in two dimensions.

Following our Hamiltonian analysis in Section 3 for the first curvature theory, we found that solutions of the equations of motion are given by plane curves. From the expression (3.5) for δS it is clear that any two curves lying on the same plane and with the same initial and final conditions for $(\mathbf{x}, \dot{\mathbf{x}})$ will share the same value of S_1. This being in complete agreement with the fact that e and κ_1 are gauge degrees of freedom in the Hamiltonian analysis. Moreover, by virtue of reparametrization invariance, S_1 can only depend on $\dot{\mathbf{x}} = e\mathbf{v}_1$ through the unit vector $\mathbf{v}_1$. Now, from the additional properties of Euclidean invariance and dilatation invariance of the action we learn that S_1 can only depend on the angle between endpoint tangents. In particular, for smooth plane closed curves the integrated curvature has to be a global invariant.

We note in passing that the reinterpretation of the two-dimensional particle model in terms of the Gauss map provides us with a simple proof of "Umlaufsatz Theorem": the integral of the signed curvature on a closed path in $\mathbf{R}^2$ is always proportional to 2π times the degree of the curve. This is so because in dimension two the Gauss map for a closed curve is a map from S^1 to S^1. Therefore the integrated signed curvature is nothing but the integrated signed length in S^1 (positive when the movement is clockwise, negative otherwise) and this can only be 2π times the number of times the Gauss map wraps S^1 into S^1. Notice that Hamiltonian methods were a little short of obtaining this result because for it to make sense we have to assume that the curve is "nicely curved", *i.e.* κ_1 nowhere zero. Nevertheless a little of cutting and pasting, together with the fact that the integrated curvature varies smoothly as we go along the curve, suffice to fill the required gaps.

From the previous discussion one could naïvely expect a similar result for the integrated second curvature in $\mathbf{R}^3$, but it is well known that it varies continuously as we deform the curve. This result can also be understood in terms of the Hamiltonian formalism and the Gauss map as follows. In this case the stabilization of the three primary constraints shows that only two of them remain first-class, therefore the action is no longer "topological", as a simple counting of degrees of freedom reveals. Moreover, as we said in Section 3, the second Gauss maps will take us from S^1 into S^1 but it is no longer true that closed curves in $\mathbf{R}^3$ are mapped into closed curves in S^1 due to the nontrivial holonomy of the two-sphere.

It is also amusing to see, via Frenet equations, that the solutions for the second curvature models in $\mathbf{R}^3$ are the same as the ones of the standard "rigid particle". The three dimensional Frenet equations explicitly read

$$
\begin{aligned}
\frac{d\mathbf{v}_1}{ds} &= \kappa_1 \mathbf{v}_2, \\
\frac{d\mathbf{v}_2}{ds} &= \kappa_2 \mathbf{v}_3 - \kappa_1 \mathbf{v}_1, \\
\frac{d\mathbf{v}_3}{ds} &= -\kappa_2 \mathbf{v}_2.
\end{aligned}
\tag{5.1}
$$

Notice that the third equation allows us to reinterpret κ_2 as the modulus of the velocity of change of the binormal ($\mathbf{v}_2$). One can now easily show, as before, that the second curvature model is equivalent to a relativistic particle on the sphere, where now the Gauss map is

defined with the binormal. Therefore solutions to the equations of motion are mapped into geodesics in S^2 or equivalently the binormal lies in a plane, but then it is automatically implied by (5.1) that $\mathbf{v}_2$ and $\mathbf{v}_1$ lie on the same plane thus giving us a contradiction. From all of this it follows that $\mathbf{v}_3$ must be zero and solutions of the equations of motion must lie on a plane, which are clearly global minima of the action.

Our constraint analysis in Section 3 implies that solutions of the equations of motion are given by curves lying on four-dimensional hyperplanes which satisfy the curvature constraint $\kappa_1 = \kappa_3$. In addition, the einbein e and the curvatures κ_1 and κ_2 turn out to be pure gauge degrees of freedom. Consequently, any two curves satisfying $\kappa_1 = \kappa_3$, with the same conditions on $(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ at the endpoints, will share the same value of S_2. We conclude from this that the integral of the second curvature provides us with a global invariant for curves in four dimensions subject to the above curvature constraint.

A similar analysis to the one performed for the first integrated curvature shows that due to Euclidean and dilatation invariance the integrated second curvature can only depend on the angles rotated by $\mathbf{v}_1$ and $\mathbf{v}_2$ as we move along the curve. It seems reasonable to conjecture that this global invariant measures the rotation index of $\mathbf{v}_2$, although we have not yet a proof of such statement.

It is left as an open problem to determine whether the integrated higher order curvatures provide us with global invariants of curves in higher dimensions subject to curvature constraints.

§6   Some Open Problems

We hope to have convinced the reader who has come this far that the particle models under study have both a rich algebraic and geometric structure. We think, therefore, that an interesting open challenge remains in studying the quantum field theories which are naturally associated with them. We expect that the analogy with free relativistic particles living on lower dimensional spheres could provide the adequate technical tools to meet that challenge.

Until now we have seen what particle models can do for W-algebras, but we still do not know what W-algebras can do for particle models. It is a tantalizing possibility that the problem of classification of W-algebras is related to the classification problem of these geometrical particle models with extended gauge symmetry. If so, it is only natural to expect that the connection of semi-simple Lie algebras with W-algebras, via the Drinfeld-Sokolov formalism, should also be relevant to the particle case. An interesting enough possibility to deserve further study.

It will also be interesting to study whether our geometric picture of W-symmetry in particle models could be extended to the string case. If that is the case it would be possible to obtain covariant extensions of W-algebras, to understand the geometry behind W-gravity, and also to propose an alternative candidate to Polyakov's rigid string for the description of QCD in four dimensions. Work on this is in progress.

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